REPRESENTATION STABILITY IN THE (CO)HOMOLOGY OF VERTICAL CONFIGURATION SPACES

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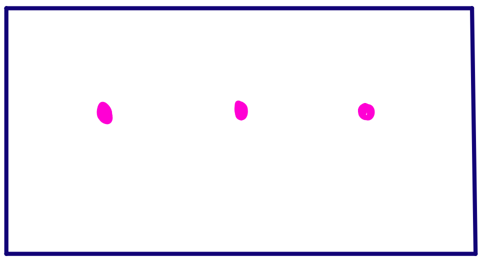
(Un)ordered Configuration Spaces

$$\mathrm{Conf}_n(M) = \{ (x_1, \dots, x_n) \in M^n,\ x_i \neq x_j \}$$

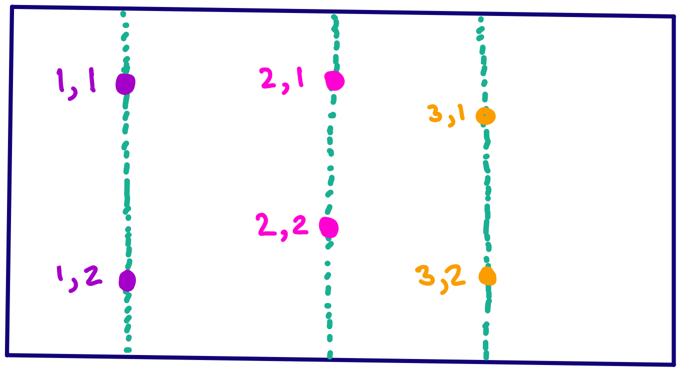
$\in \mathrm{Conf}_3 \mathbb{R}^2$

$$\mathrm{UConf}_n(M) = \mathrm{Conf}_n(M) / S_n$$

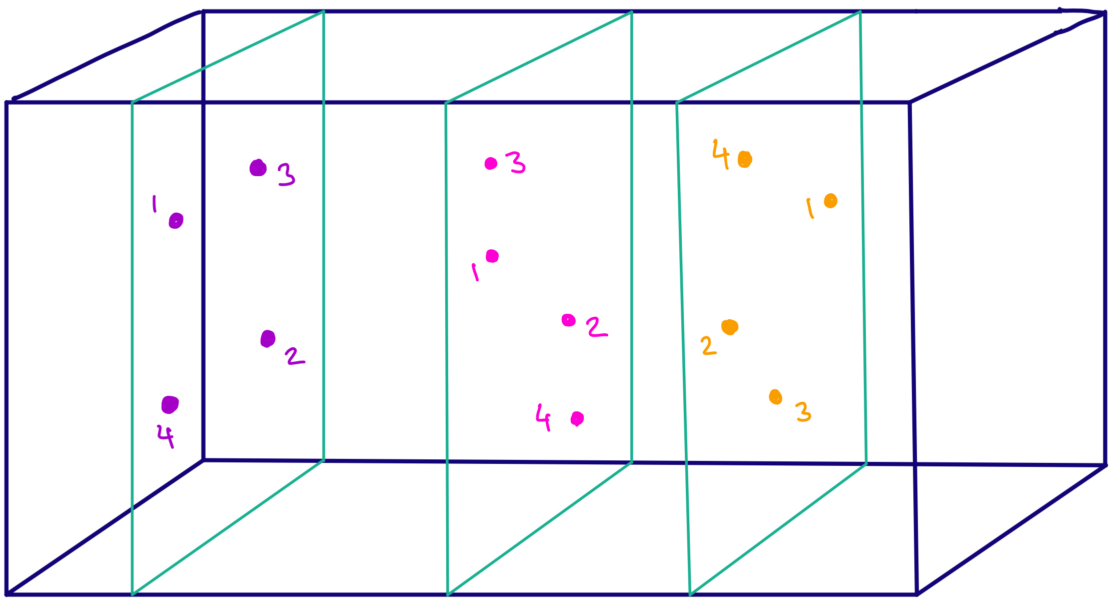


$\in \mathrm{UConf}_3 \mathbb{R}^2$

Vertical Configuration Spaces



$$\in \tilde{V}^2_3(\mathbb{R}^{1+1})$$

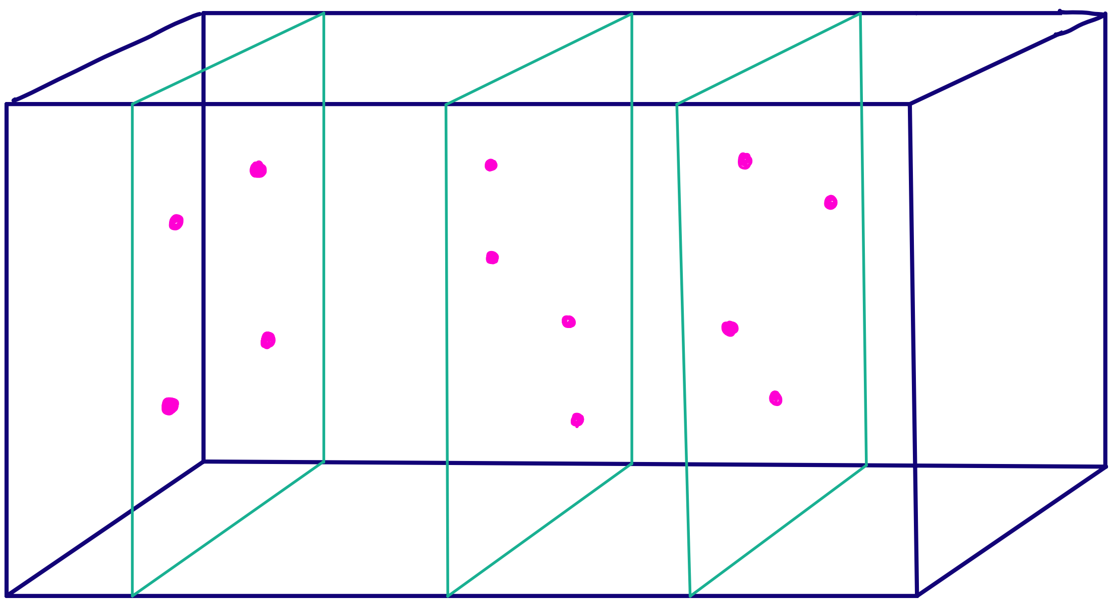


$$\in \tilde{V}^4_3(\mathbb{R}^{1+2})$$

$\tilde{V}^k_n(\mathbb{R}^{p+q})$: n clusters of k points each ; each cluster has the same first p coords.

$S_k \wr S_n \curvearrowright \tilde{V}^k_n(\mathbb{R}^{p+q})$

Unordered Version : $V^k_n(\mathbb{R}^{p+q}) = \tilde{V}^k_n(\mathbb{R}^{p+q}) / S_k \wr S_n$



$\in V^4_3(\mathbb{R}^{1+2})$

Goal: Study $H_*(V_n^k(\mathbb{R}^{p+q});\mathbb{Q})$ and $H_*(\tilde{V}_n^k(\mathbb{R}^{p+q});\mathbb{Q})$ as $S_k \wr S_n$-representations

Thm [Baron - P. - Wang - Wilson - Yang]

Fix k, d.

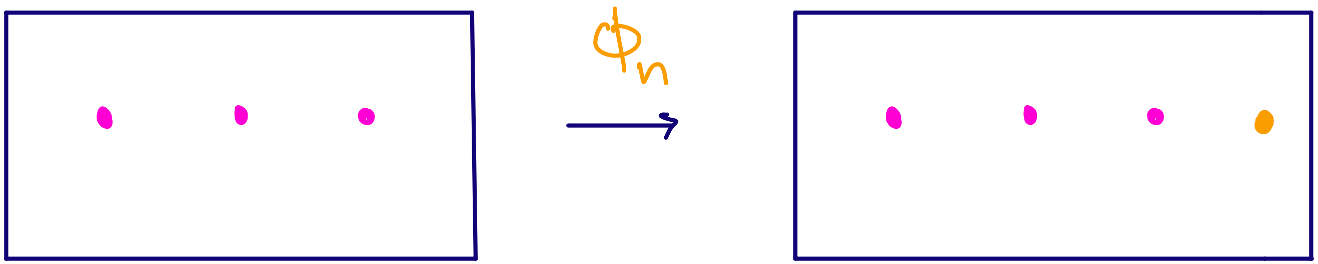
$\{H_d(\tilde{V}_n^k(\mathbb{R}^{p+q});\mathbb{Q})\}_n$ is uniformly representation stable as a $S_k \wr S_n$-representation, with stable range $n \geq \lfloor \frac{4d}{q-1} \rfloor$

Assume $q \geq 2$

Homological & Representation Stability

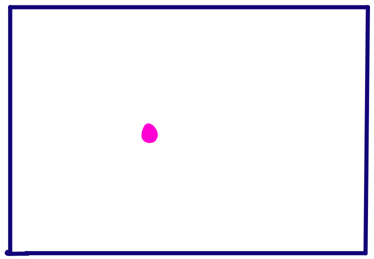
Will stick to ordinary conf. spaces ...

Have natural maps $\mathrm{UConf}_n \mathbb{R}^2 \to \mathrm{UConf}_{n+1} \mathbb{R}^2$

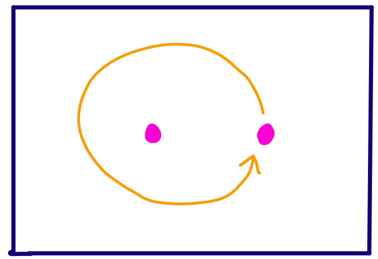


$\rightsquigarrow$ induce isos on $H_d(\mathrm{UConf}_n \mathbb{R}^2)$ for $n \geq 2d$

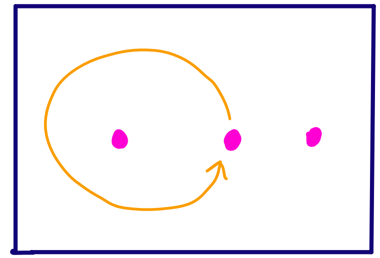
H_1 :



$n = 1$

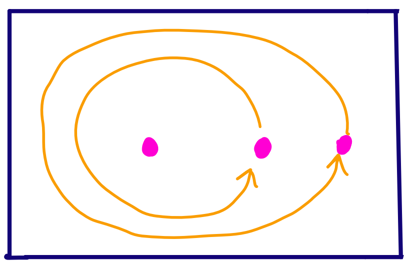


$n = 2$

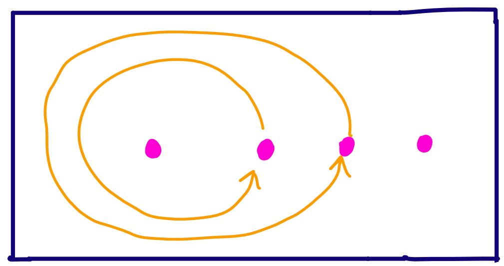


$n = 3$

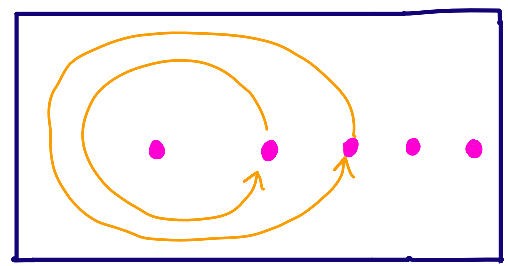
H_2 :



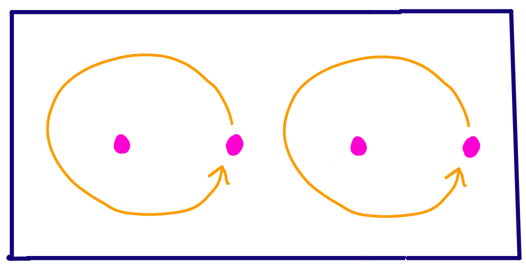
$n = 3$

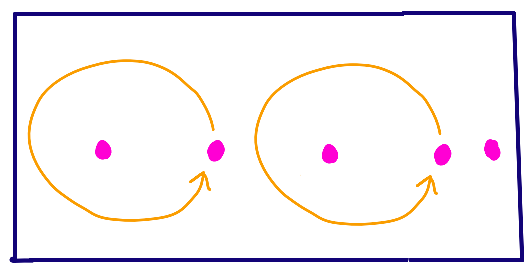


$n = 4$



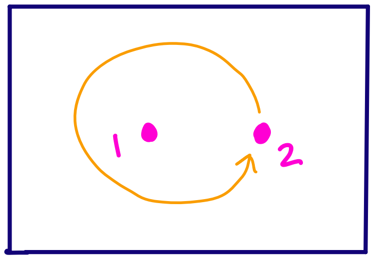
$n = 5$



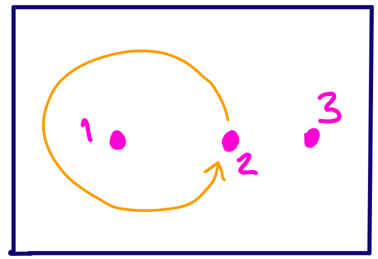


Hom. stability breaks for ordered conf. spaces ...

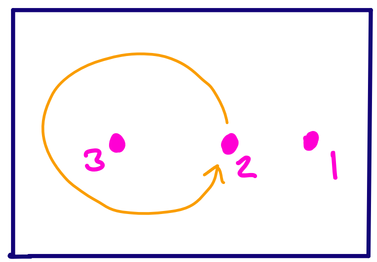
H_1 :



$n = 2$



$n = 3$



<u>Idea of rep. stability</u> : "labels are the only barrier to homological stability"

- Church - Farb formulated a notion of representation stability for a sequence of S_n - representations

(Eg: $S_n \curvearrowright H_d(\mathrm{Conf}_n \mathbb{R}^2)$)

- Eg: $V_n = H_1(\mathrm{Conf}_n \mathbb{R}^2)$

$\vdots$

$n = 4$ $\qquad V_{\ydiagram{4}} \oplus V_{\ydiagram{3,1}} \oplus V_{\ydiagram{2,2}}$

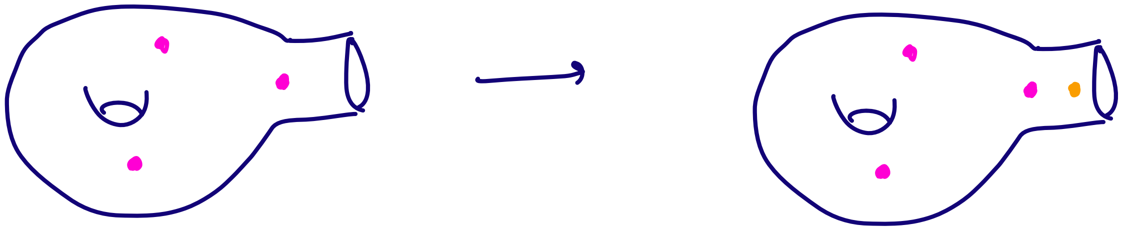
$n = 5$ $\qquad V_{\ydiagram{5}} \oplus V_{\ydiagram{4,1}} \oplus V_{\ydiagram{3,2}}$

$n = 6$ $\qquad V_{\ydiagram{6}} \oplus V_{\ydiagram{5,1}} \oplus V_{\ydiagram{4,2}}$

$\vdots$

Thm [McDuff, Segal, '70s]

dim $M \geq 2$, M not closed



$UConf_n M \longrightarrow UConf_{n+1} M$ induces isos on

H_d for $n \geq 2d$

Thm [Church, 2013]

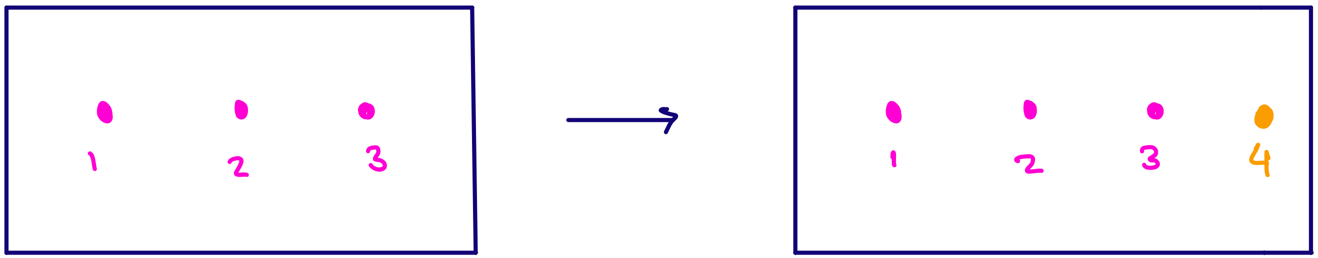
dim $M \geq 2$.

$\{H_d(Conf_n M)\}_n$ is rep. stable for $n \geq 4d$.

QUESTION : HOW DO WE PROVE REPRENTATION STABILITY ?

We have S_n-equivariant maps

$$\overset{S_1 \circlearrowright}{\mathrm{Conf}_1 \mathbb{R}^2} \longrightarrow \overset{S_2 \circlearrowright}{\mathrm{Conf}_2 \mathbb{R}^2} \longrightarrow \overset{S_3 \circlearrowright}{\mathrm{Conf}_3 \mathbb{R}^2} \longrightarrow \overset{S_4 \circlearrowright}{\mathrm{Conf}_4 \mathbb{R}^2} \longrightarrow \cdots$$



adding a "point at infinity"

- Church - Ellenberg - Farb packaged the existence and compatibility of such maps into the concept of an $\boxed{\text{FI - module}}$

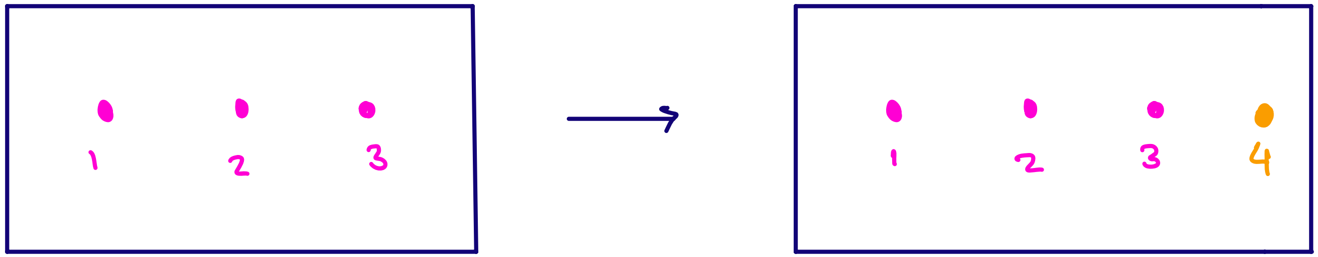
- Representation Stability $\longleftrightarrow$ finitely generated FI-module

- Deduced structural results on finitely-generated FI-modules.

- Finite-generation $\Longrightarrow$ polynomiality of rank V_n = rank $H_d(\mathrm{Conf}_n \mathbb{R}^d)$

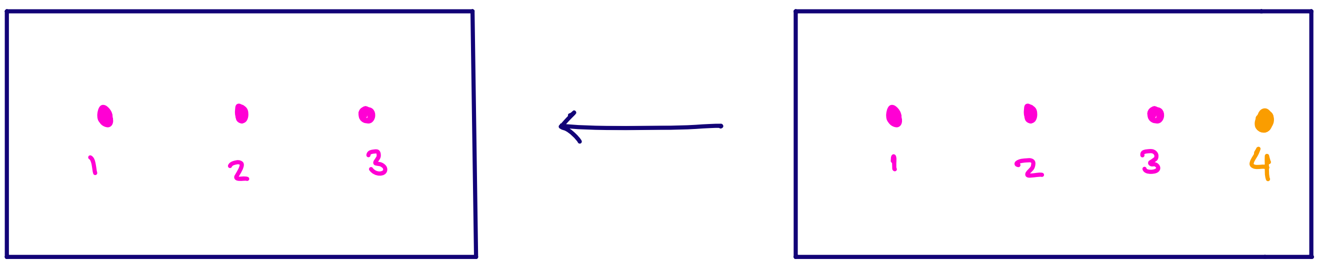
We have S_n-equivariant maps

① $$\begin{tikzcd} \mathrm{Conf}_1\mathbb{R}^2 \arrow[loop above, "S_1"] \arrow[r] & \mathrm{Conf}_2\mathbb{R}^2 \arrow[loop above, "S_2"] \arrow[r] & \mathrm{Conf}_3\mathbb{R}^2 \arrow[loop above, "S_3"] \arrow[r] & \mathrm{Conf}_4\mathbb{R}^2 \arrow[loop above, "S_4"] \arrow[r] & \cdots \end{tikzcd}$$



adding a "point at infinity"

② $$\begin{tikzcd} \mathrm{Conf}_1\mathbb{R}^2 \arrow[loop above, "S_1"] & \mathrm{Conf}_2\mathbb{R}^2 \arrow[loop above, "S_2"] \arrow[l] & \mathrm{Conf}_3\mathbb{R}^2 \arrow[loop above, "S_3"] \arrow[l] & \mathrm{Conf}_4\mathbb{R}^2 \arrow[loop above, "S_4"] \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$



forget the last point

- Church - Ellenberg - Farb packaged these into the concept of an $\boxed{\text{FI}\# \text{ - module}}$

- Deduced structural results on finitely-generated FI# - modules.

- Finite-generation $\longleftrightarrow$ polynomiality of rank V_n = rank $H_d(\mathrm{Conf}_n \mathbb{R}^d)$

Using formalism of FI_G-modules [Sam - Snowden, Gan - Li, Casto, Ramos]

Thm [Baron - P. - Wang - Wilson - Yang]

Assume $q \geq 2$

$H_d(\tilde{V}^k_n(\mathbb{R}^{p+q}); \mathbb{Q})$ is uniformly representation stable as a $S_k \wr S_n$ - representation, with stable range $n \geq \left\lfloor \frac{4d}{q-1} \right\rfloor$

Cor: $H_d(V^k_n(\mathbb{R}^{p+q}); \mathbb{Q})$ is homologically stable, with stable range $n \geq \left\lfloor \frac{2d}{q-1} \right\rfloor$

(Already known w/ $\mathbb{Z}$-coeffs, stable range $n \geq 2d$) [Tran, Palmer, Latifi, Bianchi-Kranhold]

THANK YOU!